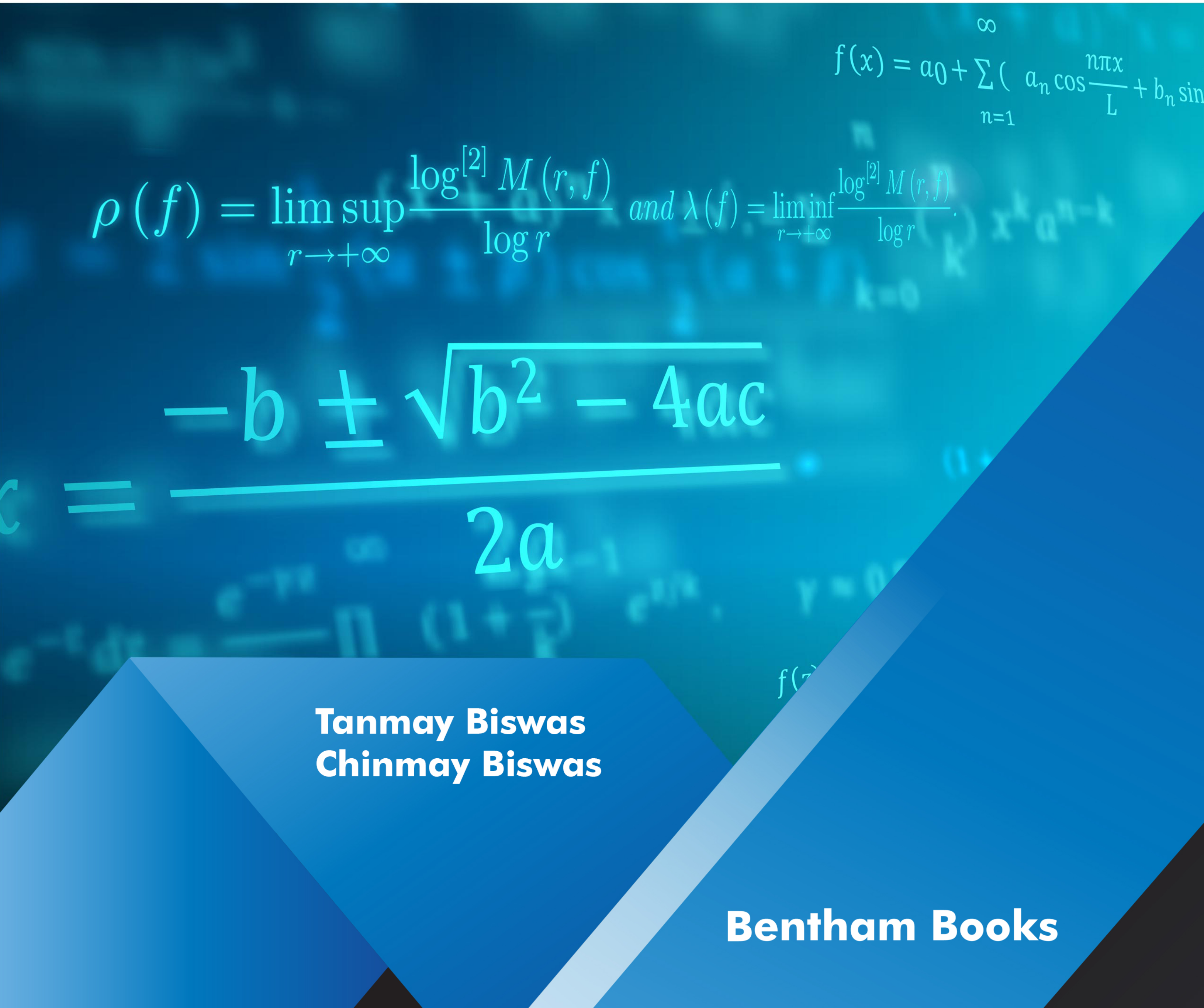
ON GENERALIZED GROWTH RATES OF INTEGER TRANSLATED ENTIRE AND MEROMORPHIC FUNCTIONS



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ISBN (Online): 978-981-5123-61-6

ISBN (Print): 978-981-5123-62-3

ISBN (Paperback): 978-981-5123-63-0

First published in 2023.

Bentham Science Publishers Pte. Ltd.
80 Robinson Road #02-00
Singapore 068898
Singapore
Email: subscriptions@benthamscience.net

Contents

PREFACE

The aim of this monograph is to discuss in detail about the growth properties of integer translated entire and meromorphic functions on the basis of their (*p; q; t*) *L*-order and (*p; q; t*) *L*-type. This book contains six chapters where we step by step elaborate the topic.

Chapter 1 contains the preliminary definitions and notations. In **Chapter 2** and **Chapter 3**, we have derived some results related to (*p; q; t*) *L*-th order and (*p; q; t*) *L*-th lower order of composite entire and meromorphic functions on the basis of their integer translation. In **Chapter 4**, we have established some relations of integer translated composite entire and meromorphic functions on the basis of their (*p; q; t*) *L*-th type and (*p; q; t*) *L*-th weak type. **Chapter 5** deals with some results about (*p; q; t*) *L*-th order and (*p; q; t*) *L*-th type of composite entire and meromorphic functions on the basis of their integer translation. **Chapter 6** is focused on some results about (*p; q; t*) *L*-th order and (*p; q; t*) L-th type of composite entire and meromorphic functions on the basis of their integer translation.

We are thankful to the authors whose publications help us to develop the results of this monograph. We think this monograph will be very helpful for future researchers and students. We are also grateful to Bentham Science Publishers for giving us the opportunity to publish this monograph.

CONSENT FOR PUBLICATION

Not applicable.

CONFLICT OF INTEREST

The authors declare no conflict of interest, financial or otherwise.

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ACKNOWLEDGEMENTS

The authors of the present book are very much thankful to the authors of different publications as many new ideas are created from them, and their publications are arranged in the Bibliography. The authors are highly thankful to Bentham Science Publishers for providing the opportunity to publish this book and very much thankful to Ms. Asma Ahmed for her kind correspondence. Authors also express gratefulness to their family members for their continuous help, inspiration, encouragement and sacrifices, lack of which the book cannot be executed. The authors will remain ever grateful to those who helped by giving constructive suggestions so that the work got a good finish. The authors are also responsible for any possible errors and shortcomings, whatever they may be in the book, despite the best attempt to make it immaculate.

(The Authors)

CHAPTER 1

Preliminary Definitions and Notations

Abstract: In this chapter, we discuss the introductory parts connected to the growth of entire and meromorphic functions and some definitions relating to the growth indicators such as order, type, generalized order, generalized type, m-th generalized ${}_pL^*$-order, m-th generalized ${}_pL^*$-type, $(p,q,t)L$-th order, $(p,q,t)L$-th type.

Keywords: Entire function, meromorphic function, generalized order, generalized type, m-th generalized ${}_pL^*$-order, m-th generalized ${}_pL^*$-type, $(p,q,t)L$-th order, $(p,q,t)L$-th type.
Mathematics Subject Classification (2020) : 30D30, 30D35.

1.1 Introduction

Let us consider that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's theory of meromorphic functions, which are available in [1-3]. We also use the standard notations and definitions of the theory of entire functions, which are available in [4, 5]. Some related basic theories of entire and meromorphic functions are briefly discussed in [6, 7], so here we do not repeat those.

Throughout this monograph, we consider that $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the sets of positive integers. We define $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$.

1.2 Preliminary Definitions and Notations

Considering above, the following definitions are relevant and have been frequently used in the monograph.

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Definition 1.2.1 *The order $\rho(f)$ and the lower order $\lambda(f)$ of an entire function $f(z)$ are defined as*

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log^{[2]}M(r,f)}{\log r} \text{ and } \lambda(f)=\liminf_{r\to+\infty}\frac{\log^{[2]}M(r,f)}{\log r}.$$

For meromorphic $f(z)$,

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r} \text{ and } \lambda(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

Next, to compare the growth of entire or meromorphic functions having the same order, one may give the definitions of type and lower type in the following manner:

Definition 1.2.2 *The type $\sigma(f)$ and the lower type $\overline{\sigma}(f)$ of an entire function $f(z)$ are defined as*

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log M(r,f)}{r^{\rho(f)}} \text{ and } \overline{\sigma}(f)=\liminf_{r\to+\infty}\frac{\log M(r,f)}{r^{\rho(f)}},$$

where $0<\rho(f)<\infty$.

If $f(z)$ is meromorphic, then

$$\sigma(f)=\limsup_{r\to+\infty}\frac{T(r,f)}{r^{\rho(f)}} \text{ and } \overline{\sigma}(f)=\liminf_{r\to+\infty}\frac{T(r,f)}{r^{\rho(f)}},$$

where $0<\rho(f)<\infty$.

It is obvious that $0\le\overline{\sigma}(f)\le\sigma(f)\le\infty$.

Likewise, to compare the growth of entire or meromorphic functions having the same lower order, one may give the definitions of upper weak type and weak type in the following manner:

Definition 1.2.3 [8] *The upper weak type $\tau(f)$ and the weak type $\overline{\tau}(f)$ of an entire function $f(z)$ of finite positive lower order $\lambda(f)$ are defined by*

$$\tau(f)=\limsup_{r\to+\infty}\frac{\log M(r,f)}{r^{\lambda(f)}} \text{ and } \overline{\tau}(f)=\liminf_{r\to+\infty}\frac{\log M(r,f)}{r^{\lambda(f)}},$$

where $0<\lambda(f)<\infty$.

If $f(z)$ is meromorphic, then

$$\tau(f)=\limsup_{r\to+\infty}\frac{T(r,f)}{r^{\lambda(f)}} \text{ and } \overline{\tau}(f)=\liminf_{r\to+\infty}\frac{T(r,f)}{r^{\lambda(f)}},$$

where $0<\lambda(f)<\infty$.

It is obvious that $0\le\overline{\tau}(f)\le\tau(f)\le\infty$.

Definition 1.2.4 *The hyper order $\overline{\rho}(f)$ and the hyper lower order $\lambda(f)^{-}$ of an entire function $f(z)$ are defined as*

$$\overline{\rho}(f) = \limsup_{r\to+\infty} \frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \overline{\lambda}(f) = \liminf_{r\to+\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

When $f(z)$ is meromorphic, then

$$\overline{\rho}(f) = \limsup_{r\to+\infty} \frac{\log^{[2]} T(r,f)}{\log r} \text{ and } \overline{\lambda}(f) = \liminf_{r\to+\infty} \frac{\log^{[2]} T(r,f)}{\log r}$$

holds.

The following two definitions are the natural consequences of the above study:

Definition 1.2.5 *The hyper type $\widehat{\sigma}(f)$ and the hyper lower type $\widehat{\overline{\sigma}}(f)$ of an entire function $f(z)$ are defined as*

$$\widehat{\sigma}(f) = \limsup_{r\to+\infty} \frac{\log^{[2]} M(r,f)}{r^{\overline{\rho}(f)}} \text{ and } \widehat{\overline{\sigma}}(f) = \liminf_{r\to+\infty} \frac{\log^{[2]} M(r,f)}{r^{\overline{\rho}(f)}},$$

where $0 < \overline{\rho}(f) < \infty$.

If $f(z)$ is meromorphic, then

$$\widehat{\sigma}(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{r^{\overline{\rho}(f)}} \text{ and } \widehat{\overline{\sigma}}(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{r^{\overline{\rho}(f)}},$$

where $0 < \overline{\rho}(f) < \infty$.

It is obvious that $0 \le \widehat{\overline{\sigma}}(f) \le \widehat{\sigma}(f) \le \infty$.

Definition 1.2.6 *The hyper upper weak type $\widehat{\tau}(f)$ and the hyper weak type $\widehat{\overline{\tau}}(f)$ of an entire function $f(z)$ of finite positive hyper lower order $\overline{\lambda}(f)$ are defined by*

$$\widehat{\tau}(f) = \limsup_{r\to+\infty} \frac{\log^{[2]} M(r,f)}{r^{\overline{\lambda}(f)}} \text{ and } \widehat{\overline{\tau}}(f) = \liminf_{r\to+\infty} \frac{\log^{[2]} M(r,f)}{r^{\overline{\lambda}(f)}},$$

where $0 < \overline{\lambda}(f) < \infty$.

If $f(z)$ is meromorphic, then

$$\widehat{\tau}(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{r^{\overline{\lambda}(f)}} \text{ and } \widehat{\overline{\tau}}(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{r^{\overline{\lambda}(f)}},$$

where $0 < \overline{\lambda}(f) < \infty$.

It is obvious that $0 \le \widehat{\overline{\tau}}(f) \le \widehat{\tau}(f) \le \infty$.

CHAPTER 2

$(p, q, t)L$-th Order Oriented Some Growth Analysis of Composite Entire and Meromorphic Functions on the Basis of Their Integer Translation

Abstract: The main objective of this chapter is to investigate some results related to the growth rates of the composition of integer translated entire and meromorphic functions using $(p, q, t)L$-th order and $(p, q, t)L$-th lower order.

Keywords: Integer translated entire function, Integer translated meromorphic function, composition, $(p, q, t)L$-th order, $(p, q, t)L$-th lower order.
Mathematics Subject Classification (2020) : 30D30, 30D35.

2.1 Introduction

Let $f(z)$ be a meromorphic function and $n \in \mathbb{N}$, then the translation of $f(z)$ be denoted by $f(z+n)$. For each $n \in \mathbb{N}$, one may obtain a function with some properties. Let us consider this family by $f_n(z)$ where

$$f_n(z) = \{f(z+n) : n \in \mathbb{N}\}.$$

We recall that if α is a regular point of an analytic function $f(z)$ and if $f(\alpha) = 0$, then α is called a zero of $f(z)$. The point $z = \alpha$ is called a zero of $f(z)$ of multiplicity m (m being a positive integer) if in some neighborhood of α, $f(z)$ can be expanded in a Taylor's series of the form $f(z) = \sum_{n=m}^{\infty} a_n (z-\alpha)^n$ where $a_m = 0$.

It is clear that the number of zeros of $f(z)$ may be changed in a finite region

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after translation but it remains unaltered in the open complex plane $\mathbb{C}$, *i.e.,*

$$N(r, f(z+n)) = N(r, f) + e_n, \tag{2.1.1}$$

where e_n is a residue term such that $e_n \to 0$ as $r \to +\infty$.

Also

$$m(r, f(z+n)) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f(re^{i\theta} + n) \right| d\theta$$
$$i.e., m(r, f(z+n)) = m(r, f) + e_n', \tag{2.1.2}$$

where e_n' (may be distinct from e_n) be such that $e_n' \to 0$ as $r \to +\infty$.

Therefore from (2.1.1) and (2.1.2), one may obtain that

$$N(r, f(z+n)) + m(r, f(z+n)) = N(r, f) + e_n + m(r, f) + e_n'$$
$$i.e., T(r, f(z+n)) = T(r, f) + e_n + e_n'.$$

Now if n varies, then Nevanlinna's Characteristic function for the family $f_n(z)$, where $f_n(z) = \{f(z+n) : n \in \mathbb{N}\}$ for the meromorphic function f is

$$T(r, f_n) = nT(r, f) + \sum_n (e_n + e_n'). \tag{2.1.3}$$

Similarly, one can define a family for each $m \in \mathbb{N}$, $g_m(z) = \{g(z+m) : m \in \mathbb{N}\}$ where $g(z)$ is an entire function. Then the composition $f_n \circ g_m$ is defined.

Let $f_n \circ g_m = h_t$, where h is a meromorphic function and $t \in \mathbb{N}$. So h_t can be expressed as $h_t = \{h(z+t) : t \in \mathbb{N}\}$.

Then by (2.1.3)

$$T(r, h_t) = tT(r, h) + \sum_t (e_t + e_t')$$

where $e_t, e_t' \to 0$ as $r \to +\infty$.

$$i.e., T(r, f_n \circ g_m) = tT(r, f(g)) + \sum_t (e_t + e_t'). \tag{2.1.4}$$

However, in the case of any two meromorphic functions $f(z)$ and $g(z)$, the ratio $\frac{T(r,f)}{T(r,g)}$ as $r \to +\infty$ is called as the growth of $f(z)$ with respect to $g(z)$ in terms of their Nevanlinna's Characteristic functions. Further, the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders.

Somasundaram and Thamizharasi [1] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \to +\infty$ for every positive constant "a". The more generalized concepts of L-order and L-lower order of meromorphic functions are $(p,q,t)L$-th order and $(p,q,t)L$-th lower order respectively.

The principal objective of this chapter is to investigate some results related to the growth rates of the composition of integer translated entire and meromorphic functions using $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of entire and meromorphic functions.

2.2 Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.2.1 [2] *Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$ then*

$$\lim_{r\to+\infty} \frac{T(r,f_n)}{T(r,f)} = n.$$

Lemma 2.2.2 *Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$ then*

$$\rho^{(p,q,t)L}(f_n) = \rho^{(p,q,t)L}(f) \ \text{ and } \ \lambda^{(p,q,t)L}(f_n) = \lambda^{(p,q,t)L}(f).$$

Proof By Lemma 2.2.1 $\lim\limits_{r\to+\infty} \frac{\log^{[p]} T(r,f_n)}{\log^{[p]} T(r,f)}$ exists and is equal to 1.

Now,

$$\begin{aligned}
\rho^{(p,q,t)L}(f_n) &= \limsup_{r\to+\infty} \frac{\log^{[p]} T(r,f_n)}{\log^{[q]} r + \exp^{[t]} L(r)} \\
&= \limsup_{r\to+\infty} \left\{ \frac{\log^{[p]} T(r,f)}{\log^{[q]} r + \exp^{[t]} L(r)} \cdot \frac{\log^{[p]} T(r,f_n)}{\log^{[p]} T(r,f)} \right\} \\
&= \limsup_{r\to+\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r + \exp^{[t]} L(r)} \cdot \lim_{r\to+\infty} \frac{\log^{[p]} T(r,f_n)}{\log^{[p]} T(r,f)} \\
&= \rho^{(p,q,t)L}(f) \cdot 1 \\
&= \rho^{(p,q,t)L}(f).
\end{aligned}$$

In a similar manner, $\lambda^{(p,q,t)L}(f_n) = \lambda^{(p,q,t)L}(f)$.

This proves the lemma.

2.3 Main Results

In this section, we present the main results of the chapter.

Theorem 2.3.1 *Let $f(z)$ be a meromorphic function and $g(z)$ be a non constant entire function such that $0 < \lambda^{(m,q,t)L}(f(g)) \le \rho^{(m,q,t)L}(f(g)) < +\infty$ and $0 < \lambda^{(l,q,t)L}(f) \le$*

CHAPTER 3

$(p, q, t)L$-th Order Based Some Further Results of Integer Translated Composite Entire and Meromorphic Functions

Abstract: The main purpose of this chapter is to investigate some results related to the growth rates of the composition of integer translated entire and meromorphic functions using $(p, q, t)L$-th order and $(p, q, t)L$-th lower order under certain different conditions.

Keywords: Growth, Entire function, meromorphic function, Slowly increasing function, Composition, $(p, q, t)L$-th order, $(p, q, t)L$-th lower order, integer translation.
Mathematics Subject Classification (2020): 30D30, 30D35.

3.1 Introduction

Let $\mathbb{C}$ be the set of all finite complex numbers and $f(z)$ be a meromorphic function defined on $\mathbb{C}$. Somasundaram and Thamizharasi [1] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \to +\infty$ for every positive constant "a". The more generalized concept of L-order and L-lower order of meromorphic functions are $(p, q, t)L$-th order and $(p, q, t)L$-th lower order, respectively. In the chapter, we establish some new results depending on the comparative growth properties of the composition of integer translated entire and meromorphic functions using $(p, q, t)L$-th order and $(p, q, t)L$-th lower order of entire and meromorphic functions under some what different conditions.

3.2 Main Results

In this section we present the main results of the chapter.

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Theorem 3.2.1 *Let $f(z)$ be a meromorphic function and $g(z)$ be a non constant entire function such that $0 < \lambda^{(m,q,t)L}(f(g)) \leq \rho^{(m,q,t)L}(f(g)) < \infty$ and $0 < \lambda^{(l,q,t)L}(f) \leq \rho^{(l,q,t)L}(f) < \infty$. Also let f_u and g_v be integer translations of $f(z)$ and $g(z)$, respectively, for $u, v \in \mathbb{N}$. If $\exp^{[t]} L(r) = o\left\{\log^{[l]} T(r, f_u)\right\}$ as $r \to +\infty$ then*

$$\frac{\lambda^{(m,q,t)L}(f(g))}{\rho^{(l,q,t)L}(f)} \leq \liminf_{r\to+\infty} \frac{\log^{[m]} T(r, f_u(g_v))}{\log^{[l]} T(r, f_u) + \exp^{[t]} L(r)} \leq \frac{\lambda^{(m,q,t)L}(f(g))}{\lambda^{(l,q,t)L}(f)}$$
$$\leq \limsup_{r\to+\infty} \frac{\log^{[m]} T(r, f_u(g_v))}{\log^{[l]} T(r, f_u) + \exp^{[t]} L(r)} \leq \frac{\rho^{(m,q,t)L}(f(g))}{\lambda^{(l,q,t)L}(f)}.$$

Proof From Definition 1.2.22 and in view of Lemma 2.2.2, we have for all sufficiently large positive numbers of r that

$$\log^{[m]} T(r, f_u(g_v)) \geq \left(\lambda^{(m,q,t)L}(f_u(g_v)) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right)$$

$$i.e., \ \log^{[m]} T(r, f_u(g_v)) \geq \left(\lambda^{(m,q,t)L}(f(g)) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right), \tag{3.2.1}$$

$$\log^{[l]} T(r, f_u) \geq \left(\lambda^{(l,q,t)L}(f_u) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right)$$
$$i.e., \ \log^{[l]} T(r, f_u) \geq \left(\lambda^{(l,q,t)L}(f) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right), \tag{3.2.2}$$

$$\log^{[m]} T(r, f_u(g_v)) \leq \left(\rho^{(m,q,t)L}(f_u(g_v)) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right)$$

$$i.e., \ \log^{[m]} T(r, f_u(g_v)) \leq \left(\rho^{(m,q,t)L}(f(g)) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right) \tag{3.2.3}$$

and

$$\log^{[l]} T(r, f_u) \leq \left(\rho^{(l,q,t)L}(f_u) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right)$$
$$i.e., \ \log^{[l]} T(r, f_u) \leq \left(\rho^{(l,q,t)L}(f) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right). \tag{3.2.4}$$

Also for a sequence of positive numbers of r tending to infinity

$$\log^{[m]} T(r, f_u(g_v)) \leq \left(\lambda^{(m,q,t)L}(f_u(g_v)) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right)$$

$$i.e., \ \log^{[m]} T(r, f_u(g_v)) \leq \left(\lambda^{(m,q,t)L}(f(g)) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right), \tag{3.2.5}$$

$$\log^{[l]} T(r, f_u) \leq \left(\lambda^{(l,q,t)L}(f_u) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right)$$
$$i.e., \ \log^{[l]} T(r, f_u) \leq \left(\lambda^{(l,q,t)L}(f) + \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L(r)\right), \tag{3.2.6}$$

$$\log^{[m]} T\left(r, f_u(g_v)\right) \geq \left(\rho^{(m,q,t)L}(f_u(g_v)) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L\left(r\right)\right)$$

$$i.e., \ \log^{[m]} T\left(r, f_u(g_v)\right) \geq \left(\rho^{(m,q,t)L}\left(f(g)\right) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L\left(r\right)\right) \tag{3.2.7}$$

and

$$\log^{[l]} T\left(r, f_u\right) \geq \left(\rho^{(l,q,t)L}(f_u) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L\left(r\right)\right)$$

$$i.e., \ \log^{[l]} T\left(r, f_u\right) \geq \left(\rho^{(l,q,t)L}\left(f\right) - \varepsilon\right)\left(\log^{[q]} r + \exp^{[t]} L\left(r\right)\right). \tag{3.2.8}$$

Now from $(3.2.4)$ we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[l]} T\left(r, f_u\right)}{\left(\rho^{(l,q,t)L}\left(f\right) + \varepsilon\right)} \leq \log^{[q]} r + \exp^{[t]} L\left(r\right). \tag{3.2.9}$$

Now from $(3.2.1)$ and $(3.2.9)$, it follows for all sufficiently large positive numbers of r that

$$\log^{[m]} T\left(r, f_u(g_v)\right) \geq \frac{\left(\lambda^{(m,q,t)L}\left(f(g)\right) - \varepsilon\right)}{\left(\rho^{(l,q,t)L}\left(f\right) + \varepsilon\right)} \log^{[l]} T\left(r, f_u\right)$$

$$i.e., \ \frac{\log^{[m]} T\left(r, f_u(g_v)\right)}{\log^{[l]} T\left(r, f_u\right) + \exp^{[t]} L\left(r\right)} \geq \frac{\left(\lambda^{(m,q,t)L}\left(f(g)\right) - \varepsilon\right)}{\left(\rho^{(l,q,t)L}\left(f\right) + \varepsilon\right)} \cdot \frac{\log^{[l]} T\left(r, f_u\right)}{\log^{[l]} T\left(r, f_u\right) + \exp^{[t]} L\left(r\right)}$$

$$i.e., \ \frac{\log^{[m]} T\left(r, f_u(g_v)\right)}{\log^{[l]} T\left(r, f_u\right) + \exp^{[t]} L\left(r\right)} \geq \frac{\frac{\left(\lambda^{(m,q,t)L}(f(g)) - \varepsilon\right)}{\left(\rho^{(l,q,t)L}(f) + \varepsilon\right)}}{1 + \frac{\exp^{[t]} L(r)}{\log^{[l]} T(r, f_u)}}.$$

Since $\exp^{[t]} L\left(r\right) = o\left\{\log^{[l]} T\left(r, f_u\right)\right\}$ as $r \to +\infty$, it follows from above that

$$\liminf_{r \to +\infty} \frac{\log^{[m]} T\left(r, f_u(g_v)\right)}{\log^{[l]} T\left(r, f_u\right) + \exp^{[t]} L\left(r\right)} \geq \frac{\left(\lambda^{(m,q,t)L}\left(f(g)\right) - \varepsilon\right)}{\left(\rho^{(l,q,t)L}\left(f\right) + \varepsilon\right)}. \tag{3.2.10}$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from $(3.2.10)$ that

$$\liminf_{r \to +\infty} \frac{\log^{[m]} T\left(r, f_u(g_v)\right)}{\log^{[l]} T\left(r, f_u\right) + \exp^{[t]} L\left(r\right)} \geq \frac{\lambda^{(m,q,t)L}\left(f(g)\right)}{\rho^{(l,q,t)L}\left(f\right)}. \tag{3.2.11}$$

Again from $(3.2.2)$, we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[l]} T\left(r, f_u\right)}{\left(\lambda^{(l,q,t)L}\left(f\right) - \varepsilon\right)} \geq \log^{[q]} r + \exp^{[t]} L\left(r\right). \tag{3.2.12}$$

From $(3.2.5)$ and $(3.2.12)$, it follows for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]} T\left(r, f_u(g_v)\right) \leq \frac{\left(\lambda^{(m,q,t)L}\left(f(g)\right) + \varepsilon\right)}{\left(\lambda^{(l,q,t)L}\left(f\right) - \varepsilon\right)} \log^{[l]} T\left(r, f_u\right)$$

CHAPTER 4

$(p, q, t)L$-th Type and $(p, q, t)L$-th Weak Type Based Some Growth Properties of Composite Entire and Meromorphic Functions on the Basis of Their Integer Translation

Abstract: The main objective of this chapter is to investigate some results related to the growth rates of the composition of integer translated entire and meromorphic functions using $(p, q, t)L$-th type and $(p, q, t)L$-th weak type.

Keywords: Entire function, meromorphic function, $(p, q, t)L$-th type, $(p, q, t)L$-th weak type, integer translation.
Mathematics Subject Classification (2020): 30D30, 30D35.

4.1 Introduction

Let $\mathbb{C}$ be the set of all finite complex numbers and $f(z)$ be a meromorphic function defined on $\mathbb{C}$. Somasundaram and Thamizharasi [1] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly, *i.e.*, $L(ar) \sim L(r)$ as $r \to +\infty$ for every positive constant "a". The more generalized concept of L-order and L-lower order of meromorphic functions are $(p, q, t)L$-th order and $(p, q, t)L$-th lower order, respectively. In order to compare the growth of entire or meromorphic functions having the same $(p, q, t)L$-th order or $(p, q, t)L$-th lower order, one may give the definitions of $(p, q, t)L$-th type and $(p, q, t)L$-th weak type of entire or meromorphic functions. In this chapter, we establish some new results depending on the comparative growth properties of composition of the integer translated entire and meromorphic functions using $(p, q, t)L$-th type and $(p, q, t)L$-th weak type of entire and meromorphic functions.

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4.2 Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 4.2.1 *[2] Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$ then*

$$\lim_{r\to+\infty} \frac{T(r, f_n)}{T(r, f)} = n.$$

Lemma 4.2.2 *Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$ then*

$$(i)\ \ \sigma^{(p,q,t)L}(f_n) = \begin{cases} n \cdot\ \sigma^{(p,q,t)L}(f) & \text{for } p=1 \\ \sigma^{(p,q,t)L}(f) & \text{for } p>1 \end{cases}$$

and

$$(ii)\ \ \overline{\sigma}^{(p,q,t)L}(f_n) = \begin{cases} n \cdot\ \overline{\sigma}^{(p,q,t)L}(f) & \text{for } p=1 \\ \overline{\sigma}^{(p,q,t)L}(f) & \text{for } p>1. \end{cases}$$

Proof By Lemma 4.2.1 and Lemma 2.2.2, we get that

$$\begin{aligned}\sigma^{(1,q,t)L}(f_n) &= \limsup_{r\to+\infty} \frac{T(r, f_n)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho^{(1,q,t)L}(f_n)}} \\ &= \lim_{r\to+\infty} \frac{T(r, f_n)}{T(r, f)} \cdot \limsup_{r\to+\infty} \frac{T(r, f)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho^{(1,q,t)L}(f)}} \\ &= n \cdot\ \sigma^{(1,q,t)L}(f).\end{aligned}$$

Also for $p > 1$, in view of Lemma 4.2.1, $\lim\limits_{r\to+\infty} \frac{\log^{[p-1]} T(r,f_n)}{\log^{[p-1]} T(r,f)}$ exists and is equal to 1. Therefore in view of Lemma 2.2.2 we obtain that

$$\begin{aligned}\sigma^{(p,q,t)L}(f_n) &= \limsup_{r\to+\infty} \frac{\log^{[p-1]} T(r, f_n)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho^{(p,q,t)L}(f_n)}} \\ &= \lim_{r\to+\infty} \frac{\log^{[p-1]} T(r, f_n)}{\log^{[p-1]} T(r, f)} \cdot \limsup_{r\to+\infty} \frac{\log^{[p-1]} T(r, f)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho^{(p,q,t)L}(f)}} \\ &= \sigma^{(p,q,t)L}(f).\end{aligned}$$

In a similar manner,

$$\overline{\sigma}^{(p,q,t)L}(f_n) = n \cdot\ \overline{\sigma}^{(p,q,t)L}(f) \text{ for } p=1$$

$$\text{and } \overline{\sigma}^{(p,q,t)L}(f_n) = \overline{\sigma}^{(p,q,t)L}(f) \text{ otherwise.}$$

Thus the lemma follows.

Lemma 4.2.3 *Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$ then*

$$(i)\ \tau^{(p,q,t)L}(f_n) = \begin{cases} n \cdot \ \tau^{(p,q,t)L}(f) & \text{for } p = 1 \\ \tau^{(p,q,t)L}(f) & \text{for } p > 1 \end{cases}$$

and

$$(ii)\ \overline{\tau}^{(p,q,t)L}(f_n) = \begin{cases} n \cdot \ \overline{\tau}^{(p,q,t)L}(f) & \text{for } p = 1 \\ \overline{\tau}^{(p,q,t)L}(f) & \text{for } p > 1. \end{cases}$$

The proof of Lemma 4.2.3 is omitted as it can easily be carried out in the line of Lemma 4.2.2.

4.3 Main Results

In this section, we present the main results of the chapter.

Theorem 4.3.1 *Let $f(z)$ be a meromorphic function and $g(z)$ be a non constant entire function such that $0 < \overline{\sigma}^{(m,q,t)L}(f(g)) \leq \sigma^{(m,q,t)L}(f(g)) < \infty$, $0 < \overline{\sigma}^{(l,q,t)L}(f) \leq \sigma^{(l,q,t)L}(f) < \infty$, $\rho^{(m,q,t)L}(f(g)) = \rho^{(l,q,t)L}(f)$. Also let f_u and g_v be integer translations of $f(z)$ and $g(z)$, respectively, for $u, v \in \mathbb{N}$. If $f_u(g_v) = h_t$, where h is a meromorphic function and $t \in \mathbb{N}$, then*

$$\frac{t \cdot \ \overline{\sigma}^{(1,q,t)L}(f(g))}{u \cdot \ \sigma^{(1,q,t)L}(f)} \leq \liminf_{r\to+\infty} \frac{T(r, f_u(g_v))}{T(r, f_u)} \leq \frac{t \cdot \ \overline{\sigma}^{(1,q,t)L}(f(g))}{u \cdot \ \overline{\sigma}^{(1,q,t)L}(f)} \leq \limsup_{r\to+\infty} \frac{T(r, f_u(g_v))}{T(r, f_u)} \leq \frac{t \cdot \ \sigma^{(1,q,t)L}(f(g))}{u \cdot \ \overline{\sigma}^{(1,q,t)L}(f)}$$

and

$$\frac{\overline{\sigma}^{(m,q,t)L}(f(g))}{\sigma^{(l,q,t)L}(f)} \leq \liminf_{r\to+\infty} \frac{\log^{[m-1]} T(r, f_u(g_v))}{\log^{[l-1]} T(r, f_u)} \leq \frac{\overline{\sigma}^{(m,q,t)L}(f(g))}{\overline{\sigma}^{(l,q,t)L}(f)} \leq \limsup_{r\to+\infty} \frac{\log^{[m-1]} T(r, f_u(g_v))}{\log^{[l-1]} T(r, f_u)} \leq \frac{\sigma^{(m,q,t)L}(f(g))}{\overline{\sigma}^{(l,q,t)L}(f)}$$

for $m > 1$ and $l > 1$.

Proof By the procedure of establishing (2.1.4) we can express

$$T(r, f_u(g_v)) = tT(r, f(g)) + \sum_t (e_t + e_t'),$$

where $e_t, e_t' \to 0$ as $r \to +\infty$. Therefore

$$\lim_{r\to+\infty} \frac{T(r, f_u(g_v))}{T(r, f(g))} = t.$$

CHAPTER 5

$(p,q,t)L$-th Order and $(p,q,t)L$-th Type Based Some Growth Rates of Integer Translated Composite Entire and Meromorphic Functions

Abstract: In this chapter, we establish some new results depending on the comparative growth properties of the composition of integer translated entire and meromorphic functions using $(p,q,t)L$-th order and $(p,q,t)L$-th type.

Keywords: Growth rates, Integer translated entire function, Integer translated meromorphic function, $(p,q,t)L$-th order, $(p,q,t)L$-th type.
Mathematics Subject Classification (2020) : 30D30, 30D35.

5.1 Introduction

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f(z)$ be an entire function defined on $\mathbb{C}$. The maximum modulus function corresponding to entire $f(z)$ is defined as $M(r,f) = \max\{|f(z)| : |z| = r\}$. When $f(z)$ is meromorphic, $M(r,f)$ can not be defined as $f(z)$ is not analytic. In this case, one may define another function $T(r,f)$, known as Nevanlinna's Characteristic function of $f(z)$, playing the same role as maximum modulus function in the following manner:

$$T(r,f) = N(r,f) + m(r,f),$$

where the function $N(r,f)$ and $m(r,f)$ are, respectively, the enumerative function and the proximity function corresponding to $f(z)$. For further details, one may see [1]. If $f(z)$ is an entire function, then Nevanlinna's Characteristic $T(r,f)$ of $f(z)$ reduces to $m(r,f)$.

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Lakshminarasimhan [2] introduced the idea of the functions of the L-bounded index. Later Lahiri and Bhattacharjee [3] worked on the entire functions of the L-bounded index and of non uniform L-bounded index. In this chapter, we establish some new results depending on the comparative growth properties of the composition of integer translated entire and meromorphic functions using $(p,q,t)L$-th order and $(p,q,t)L$-th type. Indeed some works in this direction have also been explored in [4-6].

5.2 Lemma

In this section, we present some lemmas which will be needed in the sequel.

Lemma 5.2.1 [7] *If $f(z)$ is meromorphic and $g(z)$ is entire, then for all sufficiently large values of r,*

$$T(r, f(g)) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

5.3 Main Results

In this section, we present the main results of the chapter.

Theorem 5.3.1 *Let $f(z)$ be a meromorphic function and $g(z)$ be a non constant entire function such that $\rho^{(m,n,t)L}(g) < \lambda^{(p,q,t)L}(f) \leq \rho^{(p,q,t)L}(f) < +\infty$ where $q \geq m$. Also let f_u and g_v be integer translations of $f(z)$ and $g(z)$, respectively, for $u, v \in \mathbb{N}$. Then*

$$\lim_{r\to+\infty} \frac{\log^{[p]} T(r, f_u(g_v))}{\log^{[p-m]} T(r, f_u)} = 0,$$

when for some $\alpha < \lambda^{(p,q,t)L}(f)$,
$\exp^{[t]} L(M(r,g)) = o\{\exp^{[m-1]}[(\log^{[q-1]} r)\exp^{[t+1]} L(r)]^{\alpha}\}$ *as $r \to +\infty$.*

Proof Let $f_u \circ g_v = h_t$, where is a meromorphic function and $t \in \mathbb{N}$. So h_t can be expressed as

$$h_t = \{(z+t) : t \in \mathbb{N}\}.$$

Then by (2.1.3) we obtain

$$T(r, h_t) = tT(r,h) + \sum_t (e_t + e_t^{'}),$$

where e_t, $e_t^{'} \to 0$ as $r \to +\infty$,

$$i.e., T(r, f_u \circ g_v) = tT(r, f \circ g) + \sum_t (e_t + e_t^{'}). \tag{5.3.1}$$

Now in view of Lemma 5.2.1, (5.3.1) and the inequality $T(r,g) \leq \log M(r,g)$ {cf. [5] } we get for all sufficiently large values of r that

$$\log^{[p]} T(r, f_u(g_v)) \leq$$
$$(\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} M(r,g) + \exp^{[t]} L(M(r,g))] + O(1) \qquad (5.3.2)$$

$$i.e., \ \log^{[p]} T(r, f_u(g_v)) \leq (\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[m]} M(r,g) + \exp^{[t]} L(M(r,g))] + O(1)$$

$$i.e., \ \log^{[p]} T(r, f_u(g_v)) \leq (\rho^{(p,q,t)L}(f) + \varepsilon) \times$$
$$[\exp^{[m-1]}[(\log^{[n-1]} r) \exp^{[t+1]} L(r)]^{(\rho^{(m,n,t)L}(g)+\varepsilon)} + \exp^{[t]} L(M(r,g))] + O(1). \qquad (5.3.3)$$

Also in view of Lemma 2.2.2, we obtain for all sufficiently large values of r that

$$\log^{[p-m]} T(r, f_u) \geq \exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f_u)-\varepsilon)}$$

$$i.e., \ \log^{[p-m]} T(r, f_u) \geq \exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}. \qquad (5.3.4)$$

Now from (5.3.3) and (5.3.4) we get for all sufficiently large values of r that

$$\frac{\log^{[p]} T(r, f_u(g_v))}{\log^{[p-m]} T(r, f_u)} \leq$$

$$\frac{(\rho^{(p,q,t)L}(f) + \varepsilon)[\exp^{[m-1]}[(\log^{[n-1]} r) \exp^{[t+1]} L(r)]^{(\rho^{(m,n,t)L}(g)+\varepsilon)}}{\exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}}$$
$$+ \frac{\exp^{[t]} L(M(r,g))] + O(1)}{\exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}}. \qquad (5.3.5)$$

Since $\rho^{(m,n,t)L}(g) < \lambda^{(p,q,t)L}(f)$, we can choose $\varepsilon(>0)$ in such a way that

$$\rho^{(m,n,t)L}(g) + \varepsilon < \ \lambda^{(p,q,t)L}(f) - \varepsilon. \qquad (5.3.6)$$

Now let for some $\alpha < \lambda^{(p,q,t)L}(f)$,
$\exp^{[t]} L(M(r,g)) = o\{\exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{\alpha}\}$ as $r \to +\infty$.

As $\alpha < \lambda^{(p,q,t)L}(f)$ we can choose $\varepsilon(>0)$ in such a way that

$$\alpha < \ \lambda^{(p,q,t)L}(f) - \varepsilon. \qquad (5.3.7)$$

Since $\exp^{[t]} L(M(r,g)) = o\{\exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{\alpha}\}$ as $r \to +\infty$ we get on using (5.3.7) that

$$\frac{\exp^{[t]} L(M(r,g))}{\exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{\alpha}} \to 0 \text{ as } r \to +\infty$$

$$i.e., \ \frac{\exp^{[t]} L(M(r,g))}{\exp^{[m-1]}[(\log^{[q-1]} r) \exp^{[t+1]} L(r)]^{(\lambda^{(p,q,t)L}(f)-\varepsilon)}} \to 0 \text{ as } r \to +\infty. \qquad (5.3.8)$$

Now in view of (5.3.5), (5.3.6) and (5.3.8) we obtain that

$$\lim_{r \to +\infty} \frac{\log^{[p]} T(r, f_u(g_v))}{\log^{[p-m]} T(r, f_u)} = 0.$$

Thus the theorem follows.

CHAPTER 6

Some Growth Properties of Integer Translated Composite Entire and Meromorphic Functions on the Basis of $(p, q, t)L$-th Order and $(p, q, t)L$-th Type

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Abstract: In this chapter, we establish some new results depending on the comparative growth properties of the composition of integer translated entire and meromorphic functions using $(p, q, t)L$-th order and $(p, q, t)L$-th type.

Keywords: Integer translated entire function, Integer translated meromorphic function, $(p, q, t)L$-th order, $(p, q, t)L$-th type.
Mathematics Subject Classification (2020) : 30D30, 30D35.

6.1 Introduction

Lakshminarasimhan [1] introduced the idea of the functions of the L-bounded index. Later Lahiri and Bhattacharjee [2] worked on the entire functions of the L-bounded index and of the non uniform L-bounded index. In this Chapter we establish some new results depending on the comparative growth properties of composition of integer translated entire or meromorphic functions using $(p, q, t)L$-th order, $(p, q, t)L$-th type and $(p, q, t)L$-th weak type.

6.2 Lemmas

In this section, we present some lemmas which will be needed in the sequel.

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Lemma 6.2.1 [3] *If $f(z)$ be meromorphic and $g(z)$ be entire then for all sufficiently large values of r,*

$$T(r, f(g)) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

6.3 Main Results

In this section, we present the main results of the chapter.

Theorem 6.3.1 *Let $f(z)$ be a meromorphic function and $g(z)$ be a non constant entire function such that $\rho^{(p,q,t)L}(f) = \rho^{(m,n,t)L}(g)$, $0 < \sigma^{(m,n,t)L}(g) < +\infty$ and $\overline{\sigma}^{(p,q,t)L}(f) > 0$ where $m - 1 = n = q$ and $p > 2$. Also let f_u and g_v be integer translations of $f(z)$ and $g(z)$, respectively, for $u, v \in \mathbb{N}$. Then*

$$\limsup_{r\to+\infty} \frac{\log^{[p]} T(r, f_u(g_v))}{\log^{[p-1]} T(r, f_u) + \exp^{[t]} L(M(r,g))}$$
$$\leq \begin{cases} \frac{\rho^{(p,q,t)L}(f)\sigma^{(m,n,t)L}(g)}{\overline{\sigma}^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M(r,g)) = o\{\log^{[p-1]} T(r, f_u)\} \\ \rho^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T(r, f_u) = o\{\exp^{[t]} L(M(r,g))\}. \end{cases}.$$

Proof Let $f_u \circ g_v = h_t$, where is a meromorphic function and $t \in \mathbb{N}$. So h_t can be expressed as

$$h_t = \{(z + t) : t \in \mathbb{N}\}.$$

Then by (2.1.3) we obtain

$$T(r, h_t) = tT(r, h) + \sum_t (e_t + e_t^{'}),$$

where e_t, $e_t^{'} \to 0$ as $r \to +\infty$,

$$i.e., T(r, f_u \circ g_v) = tT(r, f \circ g) + \sum_t (e_t + e_t^{'}). \tag{6.3.1}$$

Now in view of Lemma 6.2.1 and the inequality $T(r,g) \leq \log M(r,g)$ {cf. [4] } we get from (6.3.1) for all sufficiently large values of r

$$i.e., \ \log^{[p]} T(r, f_u(g_v)) \leqslant \log^{[p]} T_f(M(r,g)) + O(1) \tag{6.3.2}$$

$$i.e., \ \log^{[p]} T(r, f_u(g_v)) \leq$$
$$(\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[q]} M(r,g) + \exp^{[t]} L(M(r,g))] + O(1)$$
$$i.e., \ \log^{[p]} T(r, f_u(g_v)) \leq (\rho^{(p,q,t)L}(f) + \varepsilon)[\log^{[m-1]} M(r,g) + \exp^{[t]} L(M(r,g))] + O(1)$$
$$i.e., \ \log^{[p]} T(r, f_u(g_v)) \leq (\rho^{(p,q,t)L}(f) + \varepsilon) \cdot$$

$$[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(m,n,t)L}(g)}+\exp^{[t]} L(M(r,g))]+O(1).$$

Since $\rho^{(p,q,t)L}(f)=\rho^{(m,n,t)L}(g)$, we obtain from above for all sufficiently large values of r

$$i.e.,\ \log^{[p]} T(r,f_u(g_v))\le(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot$$

$$[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}+\exp^{[t]} L(M(r,g))]+O(1). \quad (6.3.3)$$

Again in view of Lemma 2.2.2, Lemma 4.2.2, we get for all sufficiently large values of r,

$$\log^{[p-1]} T(r,f_u)\ge(\overline{\sigma}^{(p,q,t)L}(f_u)-\varepsilon)[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f_u)}$$

$$i.e.,\ \log^{[p-1]} T(r,f_u)\ge(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}$$

$$i.e.,\ [\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}\le\frac{\log^{[p-1]} T(r,f_u)}{(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)}$$

$$i.e.,\ [\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}\le\frac{\log^{[p-1]} T(r,f_u)}{(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)}. \quad (6.3.4)$$

Now from (6.3.3) and (6.3.4) it follows for all sufficiently large values of r

$$\log^{[p]} T(r,f_u(g_v))\le(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\exp^{[t]} L(M(r,g))+O(1)+$$

$$(\rho^{(p,q,t)L}(f)+\varepsilon)(\sigma^{(m,n,t)L}(g)+\varepsilon)\cdot\frac{\log^{[p-1]} T(r,f_u)}{(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)}$$

$$ie.,\ \frac{\log^{[p]} T(r,f_u(g_v))}{\log^{[p-1]} T(r,f_u)+\exp^{[t]} L(M(r,g))}\le\frac{O(1)}{\log^{[p-1]} T(r,f_u)+\exp^{[t]} L(M(r,g))}$$

$$+\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)}{1+\frac{\log^{[p-1]} T(r,f_u)}{\exp^{[t]} L(M(r,g))}}+\frac{\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)(\sigma^{(m,n,t)L}(g)+\varepsilon)}{(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)}}{1+\frac{\exp^{[t]} L(M(r,g))}{\log^{[p-1]} T(r,f_u)}}. \quad (6.3.5)$$

If $\exp^{[t]} L(M(r,g))=o\{\log^{[p-1]} T(r,f_u)\}$ then from (6.3.5) we get

$$\limsup_{r\to+\infty}\frac{\log^{[p]} T(r,f_u(g_v))}{\log^{[p-1]} T(r,f_u)+\exp^{[t]} L(M(r,g))}\le\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)(\sigma^{(m,n,t)L}(g)+\varepsilon)}{(\overline{\sigma}^{(p,q,t)L}(f)-\varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above

$$\limsup_{r\to+\infty}\frac{\log^{[p]} T(r,f_u(g_v))}{\log^{[p-1]} T(r,f_u)+\exp^{[t]} L(M(r,g))}\le\frac{\rho^{(p,q,t)L}(f)\sigma^{(m,n,t)L}(g)}{\overline{\sigma}^{(p,q,t)L}(f)}$$

Again if $\log^{[p-1]} T(r,f_u)=o\{\exp^{[t]} L(M(r,g))\}$ then from (6.3.5) it follows

$$\limsup_{r\to+\infty}\frac{\log^{[p]} T(r,f_u(g_v))}{\log^{[p-1]} T(r,f_u)+\exp^{[t]} L(M(r,g))}\le(\rho^{(p,q,t)L}(f)+\varepsilon).$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above

$$\limsup_{r\to+\infty}\frac{\log^{[p]} T(r,f_u(g_v))}{\log^{[p-1]} T(r,f_u)+\exp^{[t]} L(M(r,g))}\le\ \rho^{(p,q,t)L}(f).$$

Thus the theorem is established.

SUBJECT INDEX

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Dr. Tanmay Biswas, an independent Research Scientist, took his Bachelor degree in Pure Mathematics from the University of Calcutta in 2002, Master's degree in Pure Mathematics from the University of North Bengal in 2005, and he obtained Doctoral degree from University of North Bengal in 2010. He has proved his potential in a research career, which is more than 14 years and his main focused area is the field of complex analysis. He has published seven Monographs and more than 270 research papers including 55 research papers with single authorship.

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